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Weighted Doubling Measures with Remotely Constant Weights

Santi Tasena[a], Sompong Dhompongsa[a]

[a]*Departmant of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*

Abstract. In this work, the authors define a new class of weight functions for which its associated class of weighted measures is doubling. Moreover, all weight functions of weighted doubling measures are limit functions of this class of weighted functions with uniformly bounded doubling constant. One particular implication is that all weight functions for weighted doubling measures are equivalent to semicontinuous functions.

1. Introduction

Doubling properties involves two closely related concepts: doubling spaces and doubling measures. It is well known that a complete metric space is doubling if and only if it admits a doubling measure[5] (see also [3]). In fact, a doubling space always admits infinitely many doubling measures. For example, Moschini and Tesei [6] proved that the measure $d\mu = |x|^{\lambda}dx$ is doubling on $\mathbb{R}^n$ whenever $\lambda > -n$. Their work was based on the work of Grigor'yan and Salof-Coste[2] who gave necessary and sufficient conditions for weighted measures to be doubling in case of radially weight functions. A natural question is to ask whether it is possible to classify weighted doubling measures or to find an equivalent condition for which a weighted measure is doubling. In this work, the authors study a class of weight functions which are quotient of functions with *polynomial growth in the asymptotically normal Z-direction* (see Definition Theorem 3.1) and doubling measures. It can be shown that weighted measures with this type of weight functions are doubling (see Theorem Theorem 3.2) provided that the set Z satisfies the skew condition. Moreover, all weight functions of weighted doubling measures are limits of this type of weight functions (see Proposition Theorem 3.3) with the uniformly bounded doubling constant (Theorem Theorem 3.10). As a result, the author can conclude that all weight functions of weight doubling measures are equivalent to semi-continuous weight functions (see Theorem Theorem 3.6).

Recently, Wen et al.[1] proved that every d-homogeneous measure on an Ahlfors d-regular space is mutually absolutely continuous with respect to its Ahlfors d-regular measure. As a result, every d-homogeneous measure on an Ahlfors d-regular space are mutually absolutely continuous. Since every d-homogeneous measure is absolutely continuous with respect to an Ahlfors d-regular measure, it is natural to ask the following: if a doubling measure is absolutely continuous with respect to another doubling measure, then are they actually mutually absolutely continuous? The answer to this question is affirmative (see Theorem

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Corresponding author: Santi Tasena
Email addresses: santi.tasena@cmu.ac.th (Santi Tasena), sompongd@chiangmai.ac.th (Sompong Dhompongsa)

Theorem 3.6). Note also that the above result by Wen et al.[1] had also been proved by Sjödin[8] a decade earlier and both are generalization of Jonsson[4]. Of course, these works are different in details but these points will not be specified here.

2. Preliminaries

In this work, a space means a *geodesic space*, i.e., a complete metric space for which there exists a path connecting any pair of points with length exactly equal to the distance between its endpoints. An open ball centered at $x \in X$ with radius $r \geq 0$ will be denoted by $\mathrm{B}(x, r)$. Also, a measure means a non-zero σ-finite Borel measure on a space.

Definition 2.1. *A measure ν on a space (X, ρ) is said to be* doubling on a family $\mathcal{F}$ *of balls in (X, ρ) with a doubling constant D if for any $\mathrm{B}(x, r) \in \mathcal{F}$, $\nu(\mathrm{B}(x, r)) \leq D\nu(\mathrm{B}(x, \frac{r}{2}))$. A measure ν is* doubling *if it is doubling on the family of all balls. In this case, ν is called a* doubling measure*.*

Two important families of balls considered in this work are the families of anchored balls and remote balls. The family of Z-anchored balls is the family of balls centered in Z and the family of Z-remote balls with parameter $\epsilon \in (0, 1)$ is the family of all balls $B(x, r)$ with $r \leq \epsilon\rho(x, Z)$.

Since we only consider geodesic spaces, any doubling measure is homogeneous. Here a measure ν is *α-homogeneous* if there is a constant $C > 0$ such that

$$\nu(B(x, \lambda r)) \leq C\lambda^{\alpha}\nu(B(x, r)$$

for all $x \in X$ and $r > 0$. Moreover, any doubling measure has the following well-known properties (see e.g. [7] for their proofs).

Proposition 2.2. *Assume that a measure ν is doubling with a doubling constant D. For any $x, y \in X$ and any $0 < s \leq r$,*

$$\frac{\nu(\mathrm{B}(x, r))}{\nu(\mathrm{B}(y, s))} \leq D\left(\frac{r + \rho(x, y)}{s}\right)^{\alpha} \tag{1}$$

where $\alpha = \log_2 D$.

Proposition 2.3. *Assume that a measure ν is doubling with a doubling constant D. For any $x \in X$ and any $0 < s \leq r$ such that $\mathrm{B}(x, r) \neq X$,*

$$\frac{\nu(\mathrm{B}(x, r))}{\nu(\mathrm{B}(x, s))} \geq d\left(\frac{r}{s}\right)^{\beta}. \tag{2}$$

where $d = \left(1 + D^{-1}\right)^{-1}$ and $\beta = \log_3\left(1 + D^{-1}\right)$.

Another property related to the doubling property is the volume comparison condition.

Definition 2.4. *Fixed a closed subset Z of a geodesic space (X, ρ). A measure ν on X is said to satisfy the* volume comparison condition *with respect to Z and remote parameter $\epsilon \in (0, 1)$, and with constant $V > 0$ if for any $z \in Z$ and $x \in X$ with $\rho = \rho(x, z) = \rho(x, Z) > 0$, we have $\nu(\mathrm{B}(z, r)) \leq V\nu(\mathrm{B}(x, \epsilon r/4)$.*

It follows that any doubling measure with doubling constant $D > 0$ satisfies volume comparison condition on with respect to any closed subset with the constant $V = D^4\epsilon^{-\alpha}$. Moreover, the following holds.

Proposition 2.5 ([2, Lemma 4.4]). *Fixed a closed subset Z of a geodesic space (X, ρ). Assume that a measure ν on X satisfies the volume comparison condition with constant V. If ν on X is doubling with a doubling constant $D > 0$ on the family of Z-anchored balls and Z-remote balls with remote parameter $\epsilon \in (0, 1)$, then ν is doubling with a doubling constant $D^2(D + V)$.*

Note that the above theorem is stated for manifolds in [2] but its proof does not rely on any manifold structure so it can be trivially ported to a geodesic space. Also, the doubling constant was not stated in [2, Lemma 4.4] but it can be computed directly from the proof.

Also, the assumption that ν is doubling on Z-anchored balls in Proposition Theorem 2.5 can be dropped if the set Z is fully accessible[2, Proposition 4.7]. In this work, however, we will consider a weaker condition called λ-skew condition.

Definition 2.6. *Fixed $\lambda \in (0, 1]$, a closed subset Z of a geodesic space (X, ρ) is said to satisfy λ-skew condition if for any $z \in Z$ and $r > 0$, the set $\mathrm{S}(z, r) = \mathrm{S}_\lambda(z, r) = \{x \in X \mid \lambda r \le \rho(x, z) \le \rho(x, Z) \le r\}$ is nonempty. If $\lambda = 1$, then the set Z is said to be fully accessible.*

3. Results

First, lets define functions which have polynomial growth in the asymptotically Z-normal direction. Henceforth, denote

$$\mathrm{C}(z) = \mathrm{C}_\delta(z) = \cup_{r \ge 0} \mathrm{S}_\delta(z, r) = \{x \in X \mid \delta\rho(x, z) \le \rho(x, Z)\}.$$

Definition 3.1. *Let (X, ρ) be a geodesic space, $Z \subseteq X$ be a closed set. Fix $\delta \in (0, 1)$ and $\alpha \ge \beta$. A measurable function $f : X \to [0, \infty)$ is said to have (α, β)-polynomial growth in the (δ-asymptotically) Z-normal direction if there is a constant $c \ge 1$ such that*

$$\frac{1}{c}\left(\frac{\rho(x, Z)}{\rho(y, Z)}\right)^\beta \le \frac{f(x)}{f(y)} \le c\left(\frac{\rho(x, Z)}{\rho(y, Z)}\right)^\alpha$$

for all $x, y \in \mathrm{C}_\delta(z), z \in Z$, for with $\rho(y, z) \le \rho(x, z)$.

Clearly, the family of functions which are polynomial growth in the Z-normal direction is closed under finite product, maximization, and minimization. Moreover, the functions $x \mapsto \rho(x, Z)^\alpha, \alpha \ge 0$, belongs to this family. It follows that any function which is (α, β)-polynomial growth in the Z-normal direction is equivalent to a function of the form $x \mapsto \rho(x, Z)^\beta f(x)$ where f is $(\alpha-\beta, 0)$-polynomial growth in the Z-normal direction.

If h is a polynomial growth function in the Z-normal direction, then h is Z-remotely constant in the following sense: there is a constant $C_R > 0$ depends only on $\theta \in (0, 1)$ such that $\sup_{y \in B(x,r)} h(y) \le C_R \inf_{y \in B(x,r)} h(y)$for all $x \in X - Z$ and $r \le \theta d(x, Z)$. This implies that the weighted measure $\mathrm{d}\nu_h = h\mathrm{d}\nu$ is doubling on Z-remote balls with $C_R^{-1} h(x)\nu(B(x, r)) \le \nu_h(B(x, r)) \le C_R h(x)\nu(B(x, r))$ for all $x \in X - Z$ and $r \le \theta d(x, Z)$.

To prove that ν_h is doubling for anchored balls, Z need to satisfy the λ-skew condition. Under this condition, there is a constant $C_A > 0$ depends only on $\epsilon \in (0, 1)$ such that $C_A^{-1} h(x)\nu(B(x, \epsilon r/4)) \le \nu_h(B(z, r))$ whenever $x \in \mathrm{S}_\lambda(z, r)$. If we can show that $Ch(x)\nu(B(x, \epsilon r/4)) \ge \nu_h(B(z, r))$ for some constant $C > 0$, then not only ν_h is doubling for anchored balls, it must also satisfy the volume comparison condition directly implying the doubling property of ν_h. Particularly, we have the following theorem.

Theorem 3.2. *Let (X, ρ) be a geodesic space, ν is a doubling measure on X, $Z \subseteq X$ be a closed ν-null set satisfying λ-skew condition, and $f : X \to [0, \infty)$ be $(\alpha, 0)$-polynomial growth δ-asymptotically in the Z-normal direction with $\lambda \ge \delta$. Denote $h(x) = \frac{f(x)}{\nu(B(x, \delta\rho(x, Z)))}$ for all $x \in X - Z$. Then the weighted measure $\mathrm{d}\nu_h = h\mathrm{d}\nu$ is doubling.*

Proof. Note that h also have polynomial growth δ-asymptotically in the Z-normal direction. Therefore, it remains to prove that there is a constant $C > 0$ such that $Ch(x)\nu(B(x, \epsilon r/4)) \ge \nu_h(B(z, r))$ whenever $x \in \mathrm{S}_\lambda(z, r)$. By doubling property of ν, this is equivalent to show that there is a constant $C > 0$ such that $Cf(x) \ge \nu_h(B(z, r))$ whenever $x \in \mathrm{S}_\lambda(z, r)$. However, this is follows from the fact that there is a constant $c > 0$ for which $\frac{f(x)}{f(y)} \ge c$ for all $x, y \in \mathrm{C}_\delta(z), z \in Z$ with $\rho(y, z) \le \rho(x, z)$. □

The essential ingredient in proving the above theorem is the fact that there is a constant $C > 0$ for which $C^{-1}h(x)\nu(B(x,\epsilon r/4)) \leq \nu_h(B(z,r)) \leq Ch(x)\nu(B(x,\epsilon r/4))$ whenever $x \in \mathrm{S}_\lambda(z,r)$. Clearly, any function h satisfying this must have polynomial growth in the Z-normal direction. Next, we will show that any weight function of a weighted doubling measure is a limit of this type of functions.

Henceforth, ν and $\tilde{\nu}$ are doubling measures on a space (X,ρ) with doubling constants D and $\tilde{D}$, respectively, and $Z \subseteq X$ is nonempty and closed with $\nu(Z) = \tilde{\nu}(Z) = 0$. Also, define $\mathrm{d}\nu_\kappa = h_\kappa \mathrm{d}\nu$ where

$$h_\kappa(x) = \frac{\tilde{\nu}(\mathrm{B}(x,\kappa\rho(x,Z))}{\nu(\mathrm{B}(x,\kappa\rho(x,Z))}$$

when $x \in X - Z$ and $h_\kappa(x) = 0$ when $x \in Z$. Note that ν_κ is absolutely continuous with respect to ν, so the value of h_κ on Z is actually unimportant.

If $\tilde{\nu}$ is absolutely continuous with respect to ν with the Radon-Nikodym derivative $h = \frac{\mathrm{d}\tilde{\nu}}{\mathrm{d}\nu}$. Then $h_\kappa \to h$ ν-a.e. as well as locally in L^1 when $\kappa \to 0$. This is the Lebesgue's differentiation Theorem[3, Theorem 1.8 and Theorem 2.7]. This directly implies the following.

Proposition 3.3. *If $\tilde{\nu}$ is absolutely continuous with respect to ν, then*

$$\lim_{\kappa\to 0}\int \phi \mathrm{d}\nu_\kappa = \int \phi \mathrm{d}\tilde{\nu}$$

for any $\phi \in L^\infty(X,\nu) \cap L^1(X,\nu)$.

Next, we will investigate some basic properties of measures ν_κ when κ is small. To derive precise estimates, the proof will be given in full details.

Lemma 3.4. *For any $\kappa \in (0, 1/4]$,*

$$\sup_{y\in \mathrm{B}(x,\kappa\rho(x,Z))} h_\kappa(y) \leq D^6\tilde{D}^6 \inf_{y\in \mathrm{B}(x,\kappa\rho(x,Z))} h_\kappa(y) \tag{3}$$

for all $x \in X$.

Proof. Let $x \in X$ and $r = \kappa\rho(x,Z)$. For any $y \in \mathrm{B}(x,r)$, $(1-\kappa)\rho(x,Z) \leq \rho(y,Z) \leq (1+\kappa)\rho(x,Z)$. It follows that

$$\begin{aligned}
\nu(\mathrm{B}(y,\kappa\rho(y,Z)) &\leq \nu(\mathrm{B}(y,\frac{\kappa}{1-\kappa}\rho(y,Z)) \\
&\leq D\left(\frac{\frac{\kappa}{1-\kappa}\rho(y,Z)+\rho(x,y)}{\kappa\rho(x,Z)}\right)^{\log_2 D}\nu(\mathrm{B}(x,\kappa\rho(x,Z)) \\
&\leq D\left(\frac{2}{1-\kappa}\right)^{\log_2 D}\nu(\mathrm{B}(x,\kappa\rho(x,Z)) \\
&\leq D^3\nu(\mathrm{B}(x,\kappa\rho(x,Z))
\end{aligned}$$

and

$$\begin{aligned}
\nu(\mathrm{B}(x,\kappa\rho(x,Z)) &\leq \nu(\mathrm{B}(x,\kappa(1+\kappa)\rho(x,Z)) \\
&\leq D\left(\frac{\kappa(1+\kappa)\rho(x,Z)+\rho(x,y)}{\kappa\rho(y,Z)}\right)^{\log_2 D}\nu(\mathrm{B}(y,\kappa\rho(y,Z)) \\
&\leq D\left(\frac{2+\kappa}{1-\kappa}\right)^{\log_2 D}\nu(\mathrm{B}(y,\kappa\rho(y,Z)) \\
&\leq D^3\nu(\mathrm{B}(y,\kappa\rho(y,Z))
\end{aligned}$$

Therefore,

$$D^{-3}\nu(\mathrm{B}(y,\kappa\rho(y,Z)) \le \nu(\mathrm{B}(x,\kappa\rho(x,Z)) \le D^{3}\nu(\mathrm{B}(y,\kappa\rho(y,Z)).$$

Similarly,

$$\tilde{D}^{-3}\tilde{\nu}(\mathrm{B}(y,\kappa\rho(y,Z)) \le \tilde{\nu}(\mathrm{B}(x,\kappa\rho(x,Z)) \le \tilde{D}^{3}\tilde{\nu}(\mathrm{B}(y,\kappa\rho(y,Z))$$

This means $D^{-3}\tilde{D}^{-3}h_\kappa(x) \le h_\kappa(y) \le D^3\tilde{D}^3 h_\kappa(x)$ yielding the result. □

Corollary 3.5. *If $\tilde{\nu}$ is absolutely continuous with respect to ν, then there are closed sets h_0 and h_∞ such that $h_0 = \{h = 0\}$ and $h_\infty = \{h = \infty\}$ ν-a.e.*

Proof. Pick $z \in X$ such that $\nu(\{z\}) = 0$ and let $Z = \{z\}$. Let

$$\begin{aligned}
\underline{f}(x) &= \liminf_{\kappa\to 0} \inf_{y\in \mathrm{B}(x,\kappa\rho(x,Z))} h_\kappa(y),\\
\overline{f}(x) &= \limsup_{\kappa\to 0} \inf_{y\in \mathrm{B}(x,\kappa\rho(x,Z))} h_\kappa(y),\\
\underline{g}(x) &= \liminf_{\kappa\to 0} \sup_{y\in \mathrm{B}(x,\kappa\rho(x,Z))} h_\kappa(y),\\
\overline{g}(x) &= \limsup_{\kappa\to 0} \sup_{y\in \mathrm{B}(x,\kappa\rho(x,Z))} h_\kappa(y).
\end{aligned}$$

Then $\underline{f} \le \underline{g} \le D^6\tilde{D}^6\underline{f}$, and $\overline{f} \le \overline{g} \le D^6\tilde{D}^6\overline{f}$ outside Z. Moreover, $\overline{f} \le h \le \underline{g}$ ν-a.e. It follows that $\{\underline{f} = 0\} = \{\overline{f} = 0\} = \{\underline{g} = 0\} = \{\overline{g} = 0\}$ and $\{\underline{f} = \infty\} = \{\overline{f} = \infty\} = \{\underline{g} = \infty\} = \{\overline{g} = \infty\}$. Set $A = \{\underline{f} = 0\}$ and $B = \{\underline{f} = \infty\}$. Then $A = \{h = 0\}$ and $B = \{h = \infty\}$ ν-a.e.

Let $x_n \in A$ be such that $x_n \to x \neq z$. For any $\epsilon > 0$ and $n \in \mathbb{N}$, there is a $\kappa_n > 0$ such that $\sup_{y\in \mathrm{B}(x_n,\kappa\rho(x_n,Z)} h_\kappa(y) < \epsilon$ for all $\kappa \le \kappa_n$. This follows from the fact that $\overline{g}(x_n) = 0$ for all $n \in \mathbb{N}$. For any $\kappa > 0$, choose $n_0 \in \mathbb{N}$ large enough so that $\rho(x_n, x) < \frac{\kappa}{2}\rho(x,Z)$ for all $n \ge n_0$. Thus, $\inf_{y\in \mathrm{B}(x,\kappa\rho(x,Z))} h_\kappa(y) \le \sup_{y\in \mathrm{B}(x_{n_0},\min(\frac{\kappa}{2},\kappa_{n_0})\rho(x_{n_0},Z))} h_\kappa(y) < \epsilon$. Let $\kappa \to 0$ follows by $\epsilon \to 0$, then $\overline{f}(x) = 0$, i.e., $x \in A$. Let h_0 be the closure of A. Since $h_0 \subseteq A\cup\{z\}$, $h_0 = \{h = 0\}$ ν-a.e. Similarly, define h_∞ to be the closure of B, then $h_\infty \subseteq B\cup\{z\}$ and $h_\infty = \{h = \infty\}$ ν-a.e. □

Particularly, this implies that all doubling measures absolutely continuous with respect to ν are equivalent to the ones having lower and upper semicontinuous Radon-Nikodym derivatives.

Theorem 3.6. *For any doubling measure $\tilde{\nu}$ absolutely continuous with respect to doubling measure ν with $\frac{d\tilde{\nu}}{d\nu} = h$, there exist a lower semi-continuous function $\underline{h}$ and an upper semi-continuous $\overline{h}$ such that*

(a) $\underline{h}$ and $\overline{h}$ are continuous at the same points,

(b) $\underline{h}(x) = \overline{h}(x)$ whenever $\underline{h}$ is continuous at x,

(c) $\underline{h} \le h \le D^6\tilde{D}^6\underline{h}$, and

(d) $D^{-6}\tilde{D}^{-6}\overline{h} \le h \le D^6\tilde{D}^6\overline{h}$.

Particularly, $\tilde{\nu}$ and ν are mutually absolutely continuous.

Proof. In this proof, Z be a closed set in which $Z = \{h = 0\}\cup\{h = \infty\}$ ν-a.e. Let

$$f(x) = \limsup_{\kappa\to 0} \inf_{y\in \mathrm{B}(x,\kappa\rho(x,Z))} h_\kappa(y) \text{ and}$$

$$g(x) = \liminf_{\kappa\to 0} \sup_{y\in \mathrm{B}(x,\kappa\rho(x,Z))} h_\kappa(y).$$

Then $f \le h \le g$ ν-a.e. Moreover, $g \le D^6\tilde{D}^6 f$ which implies $0 < f, g < \infty$ on $X - Z$.

To see that f is lower semi-continuous, let $x \in X$ and $r > 0$ be such that $f(x) > r$. Then for any $\kappa > 0$ such that $\inf_{y \in B(x,\kappa\rho(x,Z))} h_\kappa(y) > r$. Pick $\eta = \frac{\kappa}{2+\kappa}$. Then for any $w \in B(x, \eta\rho(x, Z))$ and any $y \in B(w, \kappa\rho(w, Z)/2)$,

$$\begin{aligned}\rho(y,x) &\le \frac{\kappa}{2}\rho(w,Z) + \rho(w,x)\\ &\le \frac{\kappa}{2}(\rho(w,x) + \rho(x,Z)) + \rho(w,x)\\ &\le \left(\left(\frac{\kappa}{2}+1\right)\eta + \frac{\kappa}{2}\right)\rho(x,Z)\\ &\le \kappa\rho(x,Z)\end{aligned}$$

It follows that $\inf_{y \in B(w,\kappa\rho(x,Z)/2)} h_{\kappa/2}(y) > r$. This proves that $f(w) > r$ for all $w \in B(x, \eta\rho(x, Z))$ immediately implying that f is lower semi-continuous. The upper semi-continuity of g can be proven similarly.

Now, let $\underline{h} = f$ and $\overline{h}(x) = \lim_{r\to 0} \sup_{y \in B(x,r)} f(y)$. Combining with the fact that $\overline{h} \le g \le D^6\tilde{D}^6 f$, *(a)-(d)* immediately follows. □

Lastly, we will show that the measures ν_κ all have the same doubling constant. The proof is based on several lemmas given below.

Lemma 3.7. *For any $\kappa \in (0, 1/4]$, the measure ν_κ is doubling with doubling constant $D^7\tilde{D}^6$ on Z-remote balls with remote parameter κ. Moreover, for all $x \in X$,*

$$D^{-3}\tilde{D}^{-3}\tilde{\nu}(B(x, \kappa\rho(x, Z))) \le \nu_\kappa(B(x, \kappa\rho(x, Z))) \le D^3\tilde{D}^3\tilde{\nu}(B(x, \kappa\rho(x, Z))). \quad (4)$$

Proof. Let $x \in X$ and $0 \le r \le \kappa\rho(x, Z)$. For any $y \in B(x, r)$,

$$D^{-3}\tilde{D}^{-3}h_\kappa(x) \le h_\kappa(y) \le D^3\tilde{D}^3 h_\kappa(x)$$

which implies $D^{-3}\tilde{D}^{-3}h_\kappa(x)\nu(B(x, r)) \le \nu_\kappa(B(x, r)) \le D^3\tilde{D}^3 h_\kappa(x)\nu(B(x, r))$. Hence,

$$\nu_\kappa(B(x, r)) \le D^4\tilde{D}^3 h_\kappa(x)\nu(B(x, \frac{1}{2}r)) \le D^7\tilde{D}^6\nu_\kappa(B(x, \frac{1}{2}r)).$$

Setting $r = \kappa\rho(x, Z)$ also implies Equation (4). □

Lemma 3.8. *For any $\kappa \in (0, 1/4]$, any $x \in X$, and any $r \ge \kappa\rho(x, Z)$,*

$$D^{-17}\tilde{D}^{-17}\tilde{\nu}(B(x, r)) \le \nu_\kappa(B(x, r)) \le D^3\tilde{D}^7\tilde{\nu}(B(x, r)). \quad (5)$$

Proof. By Whitney covering theorem, there is a countable family $\mathcal{W}$ of disjoint balls such that (a) $\cup_{B(y,s)\in\mathcal{W}} B(y, 3s) = X - Z$, and (b) $s = \frac{1}{3}\kappa\rho(y, Z)$ for all $B(y, s) \in \mathcal{W}$.

Let $x \in X$ and $r \ge \kappa\rho(x, Z)$. Let $\mathcal{B} = \{B(y, s) \in \mathcal{W} : B(y, 3s) \cap B(x, r) \ne \emptyset\}$. For any $B(y, s) \in \mathcal{B}$,

$$s = \frac{1}{3}\kappa\rho(y, Z) \le \frac{1}{3}\kappa\left(\rho(y, x) + \rho(x, Z)\right) \le \frac{1}{3}\kappa\left(r + 3s + \rho(x, Z)\right)$$

yielding $(1 - \kappa)s \le \left(\frac{1}{3}\kappa + \frac{1}{3}\right)r$ and so $s \le \left(\frac{1+\kappa}{3(1-\kappa)}\right)r \le \frac{5r}{9}$ which implies $B(y, s) \subseteq B(x, 4r)$. Therefore,

$$\begin{aligned}\nu_\kappa(B(x,r)) &\le \sum_{B(y,s)\in\mathcal{B}} \nu_\kappa(B(y,3s))\\ &\le D^3\tilde{D}^5 \sum_{B(y,s)\in\mathcal{B}} \tilde{\nu}(B(y,s))\\ &\le D^3\tilde{D}^7\tilde{\nu}(B(x,r))\end{aligned}$$

while

$$\begin{aligned}\tilde{\nu}(B(x,4r)) &\leq \tilde{D}^2\tilde{\nu}(\cup_{B(y,s)\in\mathcal{B}}B(y,3s))\\ &\leq D^3\tilde{D}^5\sum_{B(y,s)\in\mathcal{B}}\nu_\kappa(B(y,3s))\\ &\leq D^{17}\tilde{D}^{17}\sum_{B(y,s)\in\mathcal{B}}\nu_\kappa(B(y,s))\\ &\leq D^{17}\tilde{D}^{17}\nu_\kappa(B(x,4r)).\end{aligned}$$

Hence, $D^{-17}\tilde{D}^{-17}\tilde{\nu}(B(x,r)) \leq \nu_\kappa(B(x,r)) \leq D^3\tilde{D}^7\tilde{\nu}(B(x,r))$ whenever $r \geq 4\kappa\rho(x,Z)$. For $r \leq 4\kappa\rho(x,Z)$,

$$\begin{aligned}\tilde{\nu}(B(x,r)) &\leq \tilde{D}^2\tilde{\nu}(B(x,\kappa\rho(x,Z)))\\ &\leq D^3\tilde{D}^5\nu_\kappa(B(x,\kappa\rho(x,Z)))\\ &\leq D^{17}\tilde{D}^{17}\nu_\kappa(B(x,r)).\end{aligned}$$

□

Lemma 3.9. *For any $\kappa \in (0,1/4]$, $z \in Z$, and $r > 0$,*

$$D^{-17}\tilde{D}^{-16}\tilde{\nu}(B(z,r)) \leq \nu_\kappa(B(z,r)) \leq D^3\tilde{D}^6\tilde{\nu}(B(z,r)).$$

Proof. By Whitney covering theorem, there is a countable family $\mathcal{W}$ of disjoint balls such that (a) $\cup_{B(x,s)\in\mathcal{W}}B(x,3s) = X - Z$, and (b) $s = \frac{1}{3}\kappa\rho(x,Z)$ for all $B(x,s) \in \mathcal{W}$. .

Let $z \in Z$ and $r > 0$. Denote $\mathcal{B} = \{B(x,s) \in \mathcal{W} : B(x,3s)\cap B(z,r) \neq \emptyset\}$. Clearly, $B(z,r) \subseteq \cup_{B(x,s)\in\mathcal{B}}B(x,3s)$. For any $B(x,s) \in \mathcal{B}$,

$$\rho(x,Z) \leq \kappa\rho(x,Z) + r$$

implying $B(x,s) \subseteq B(z,2r)$. Using Lemma Theorem 3.7, we have

$$\begin{aligned}\nu_\kappa(B(z,r)) &\leq D^3\tilde{D}^3\sum_{B(x,s)\in\mathcal{B}}\tilde{\nu}(B(x,3s))\\ &\leq D^3\tilde{D}^5\sum_{B(x,s)\in\mathcal{B}}\tilde{\nu}(B(x,s))\\ &\leq D^3\tilde{D}^6\tilde{\nu}(B(z,r))\end{aligned}$$

while

$$\begin{aligned}\nu_\kappa(B(z,2r)) &\geq \sum_{B(x,s)\in\mathcal{B}}\nu_\kappa(B(x,s))\\ &\geq D^{-14}\tilde{D}^{-12}\sum_{B(x,s)\in\mathcal{B}}\nu_\kappa(B(x,3s))\\ &\geq D^{-17}\tilde{D}^{-15}\sum_{B(x,s)\in\mathcal{B}}\tilde{\nu}(B(x,3s))\\ &\geq D^{-17}\tilde{D}^{-16}\tilde{\nu}(B(z,2r)).\end{aligned}$$

□

Theorem 3.10. *For any $\kappa \in (0,1/4]$, the measure ν_κ is doubling with constant $D^{20}\tilde{D}^{25}$.*

Proof. Let $x \in X$ and $r > 0$. If $x \in Z$, then

$$\mu_\kappa(B(x,r)) \leq D^3\tilde{D}^6\tilde{\nu}(B(x,r)) \leq D^3\tilde{D}^7\tilde{\nu}(B(x,r/2)) \leq D^{20}\tilde{D}^{23}\nu_\kappa(B(x,r/2)).$$

If $x \notin Z$ and $r \geq \kappa\rho(x,Z)$,

$$\mu_\kappa(B(x,r)) \leq D^3\tilde{D}^7\tilde{\nu}(B(x,r)) \leq D^3\tilde{D}^8\tilde{\nu}(B(x,r/2)) \leq D^{20}\tilde{D}^{25}\nu_\kappa(B(x,r/2)).$$

Lastly, if $x \notin Z$ and $r \leq \kappa\rho(x,Z)$, $\mu_\kappa(B(x,r)) \leq D^7\tilde{D}^6\nu_\kappa(B(x,r/2))$. □

4. Conclusion and Discussion

It is well-known that doubling property is closely related to the Harnack inequality and the Gaussian behavior of heat kernel. Precisely, the parabolic Harnack inequlity is equivalent to the Gaussian behavior of heat kernel and both are equivalent to doubling property and the Poincaré inequality. Grigor'yan and Salof-Coste[2] studied whether these properties remain invariant under the change of measures. In their work, the weight functions must have polynomial growth in the asymptotically normal $\{z\}$-direction. In this work, the authors show that all weighted doubling measures are limits of weighted doubling measures of this form. Therefore, it is interesting to see whether the result of Grigor'yan and Salof-Coste[2] could be extended to any weighted doubling measures. In the authors' opinion, this should be possible since most of the proof relied only on the doubling constant and the Poincaré constant. All these require further investigation, however.

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